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VARIATION OF PARAMETERS

BY POISSON'S METHOD

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ABSTRACT

A direct derivation for the variation of the elliptic elements is given, starting from the variational equation due to Poisson. The method is suitable for any integral from elliptic motion and gives the variation of an element by the use of a single equation.

FOREWORD

This work was undertaken in support of NRL Work Request No. SPASUR WR-3-005. The purpose of this report is to give a simple method for deriving the equations of variation of constants associated with Keplerian motion.

INTRODUCTION

The well known method of variation of arbitrary constants for the solution of a linear system of differential equations is usually applied in celestial mechanics by choosing as the arbitrary constants a set of elements of the Keplerian unperturbed motion. The present memorandum shows how the same method can be used by taking the arbitrary constants equal to the first integrals of the unperturbed Keplerian motion. In some cases this leads to a simplification of the formulae needed.

Integrals of Motion

We will consider first elliptic unperturbed motion, though the method is suitable for any kind of motion under a central Newtonian attraction.

1. Energy

$$h = \frac{1}{2} \vec{v}^2 - \frac{\mu}{r}$$

2. Angular momentum

$$\vec{C} = \frac{1}{2} \vec{r} \times \vec{v}$$

3. Integral of pericenter (Laplace)

$$\vec{p} = \mu e \vec{i}$$

where $\vec{i}$ is a unit vector pointing toward pericenter, from the center of attraction.

4. Time of pericenter

$$\tau = t - \frac{\ell}{n}$$

where ℓ is the mean anomaly.

POISSON'S VARIATIONAL FORMULA

Consider a system of differential equations reduced to its canonical form:

$$\dot{x}_i = f_i(\vec{x},t) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

and let $X(\vec{x},t) = C_o$ be a first integral of this system, called "unperturbed." We are using $\vec{x}$ for the set $(x_1, x_2, \ldots, x_n)$.

Consider now the "perturbed" system:

$$\dot{x}_i = f_i(\vec{x},t) + g_i(\vec{x},t) \quad (i = 1, 2, \ldots, n) \tag{2}$$

The problem is to find the variation of C_o due to the addition of $g_i(\vec{x},t)$. Suppose, therefore

$$X(\vec{x},t) = C(t)$$

From this

$$\dot{C} = \frac{\partial X}{\partial t} + \sum_j \frac{\partial X}{\partial x_j} \dot{x}_j$$

$$= \frac{\partial X}{\partial t} + \sum_j \frac{\partial X}{\partial x_j} \left[f_j(\vec{x},t) + g_j(\vec{x},t) \right]$$

$$= \frac{dC_o}{dt} + \sum_j \frac{\partial X}{\partial x_j} g_j(\vec{x},t)$$

Therefore

$$\dot{C} = \sum_j \frac{\partial X}{\partial x_j} g_j(\vec{x},t) \tag{3}$$

Application to Celestial Mechanics

The formula (3) is now applied to the system

$$\dot{\vec{r}} = \vec{v}, \ \dot{\vec{v}} = -\frac{\mu}{r^3}\vec{r} + \vec{R} \tag{4}$$

where $\vec{R}$ is the disturbing force per unit mass. This system corresponds to six scalar differential equations in the space (x, y, z, v_x, v_y, v_z). We know first integrals when $\vec{R} = 0$.

In the case where equation (4) applies,

$$C = \varphi(\vec{r}, \vec{v}, t),$$

and

$$\dot{C} = \frac{\partial\varphi}{\partial v_x} R_x + \frac{\partial\varphi}{\partial v_y} R_y + \frac{\partial\varphi}{\partial v_z} R_z = \text{grad}_{\vec{v}}\, \varphi \cdot \vec{R},$$

so that

$$\dot{C} = \vec{R} \quad \text{grad}_{\vec{v}}\, \varphi \quad \text{(Poisson)}. \tag{5}$$

Relation (5) is quite general, C being any integral of the unperturbed motion.

APPLICATIONS

1. Integral of energy

$$h = \frac{1}{2}\vec{v}^2 - \frac{\mu}{r}$$

$$\dot{h} = \text{grad}_{\vec{v}}\left(\frac{1}{2}\vec{v}^2 - \frac{\mu}{r}\right) \cdot \vec{R} = \vec{v} \cdot \vec{R}$$

Now, in a Keplerian motion

$$h = -\frac{\mu}{2a}$$

$$\dot{h} = \frac{\mu}{2a^2}\dot{a}$$

and therefore:

$$\dot{a} = \frac{2a^2}{\mu} \vec{v} \cdot \vec{R} \tag{6}$$

2. <u>Angular momentum</u>

$$\vec{C} = \vec{r} \times \vec{v} = C\vec{k}$$

with respect to an equatorial system:

$$\vec{k} = \begin{pmatrix} \sin \Omega \sin I \\ -\cos \Omega \sin I \\ \cos I \end{pmatrix}$$

Therefore

$$\dot{\vec{C}} = \dot{C}\vec{k} + C\left(\frac{\partial \vec{k}}{\partial \Omega} \dot{\Omega} + \frac{\partial \vec{k}}{\partial I} \dot{I}\right)$$

Note that

$$\vec{k} \cdot \vec{k} = 1 \qquad \left(\frac{\partial \vec{k}}{\partial \Omega}\right)^2 = \sin^2 I$$

$$\vec{k} \cdot \frac{\partial \vec{k}}{\partial \Omega} = 0 \qquad \frac{\partial \vec{k}}{\partial \Omega} \cdot \frac{\partial \vec{k}}{\partial I} = 0$$

$$\vec{k} \cdot \frac{\partial \vec{k}}{\partial I} = 0 \qquad \left(\frac{\partial \vec{k}}{\partial I}\right)^2 = 1$$

Thus we obtain

$$\dot{\vec{C}} \cdot \vec{k} = \dot{C}$$

$$\dot{\vec{C}} \cdot \frac{\partial \vec{k}}{\partial \Omega} = C \sin^2 I \, \dot{\Omega}$$

$$\dot{\vec{C}} \cdot \frac{\partial \vec{k}}{\partial I} = C \, \dot{I}$$

Applying (5)

$$\dot{\vec{C}} = \vec{r} \times \vec{R}$$

and then

$$\dot{C} = (\vec{r} \times \vec{R}) \cdot \vec{k} = [\vec{r}, \vec{R}, \vec{k}] \tag{7}$$

$$\dot{\Omega} = \frac{1}{C \sin^2 I} \left[\vec{r}, \vec{R}, \frac{\partial \vec{k}}{\partial \Omega} \right] \tag{8}$$

$$\dot{I} = \frac{1}{C} \left[\vec{r}, \vec{R}, \frac{\partial \vec{k}}{\partial I} \right] \tag{9}$$

If we consider

$$C = \sqrt{\mu a(1-e^2)}$$

then

$$\dot{e} = \frac{1-e^2}{2ae} \dot{a} - \frac{C}{\mu ae} \dot{C}$$

Using (6) and (7), there results

$$\dot{e} = \frac{1}{\mu ae} \left\{ a^2 (1-e^2) \vec{v} \cdot \vec{R} - C \left[\vec{r}, \vec{R}, \vec{k} \right] \right\} \tag{10}$$

3. Integral of Pericenter

Let us deduce $\vec{p}$ as function of $\vec{r}$, $\vec{v}$. Consider the quantity

$$\vec{C} \times \dot{\vec{v}}$$

From

$$\dot{\vec{v}} = -\frac{\mu}{r^3}\vec{r}, \quad \vec{r} \times \vec{v} = \vec{C}$$

there results

$$\vec{C} \times \dot{\vec{v}} = \frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{v}) = \frac{\mu}{r^3}\left[(\vec{r} \cdot \vec{v})\,\vec{r} - \vec{r}^{\,2}\,\vec{v}\right]$$

$$= \mu\left[\frac{\dot{r}\,\vec{r} - \vec{r}\,\vec{v}}{r^2}\right] = -\frac{d}{dt}\left(\mu\frac{\vec{r}}{r}\right) = \frac{d}{dt}\,(\vec{C} \times \vec{v})$$

Therefore

$\vec{C} \times \vec{v} = -\mu\frac{\vec{r}}{r} - \vec{p}$, where $\vec{p}$ is a constant of integration.

If $\vec{k} \times \equiv i$, then

$\vec{C} \times \vec{v} = C\, i\, \vec{v}\ \therefore$

$$\vec{v} = \frac{i}{C}\left\{\mu\frac{\vec{r}}{r} + \vec{p}\right\} \qquad \text{(Hodograph)}$$

Let us show that

$$\vec{p} = \mu\, e\, \vec{i}\,.$$

$$-\vec{p} = \vec{C} \times \vec{v} + \mu\frac{\vec{r}}{r} = (\vec{r} \times \vec{v}) \times \vec{v} + \mu\frac{\vec{r}}{r},$$

$$\vec{p} = (\vec{v} \cdot \vec{v})\,\vec{r} - (\vec{r} \cdot \vec{v})\,\vec{v} - \mu\frac{\vec{r}}{r} \tag{11}$$

If $\vec{p}$ is constant, it may be evaluated at any point in the orbit, and we will choose the pericenter. Then

$$\vec{p} = v_p^2 \, r_p \, \vec{i} - \mu \, \vec{i} = (v_p^2 \, r_p - \mu) \, \vec{i} = \mu \, e \, \vec{i}$$

as we had to prove.

Using (11)

$$\vec{p} = (\vec{v} \cdot \vec{v}) \, \vec{r} - (\vec{v} \cdot \vec{r}) \, \vec{v} - \mu \, \frac{\vec{r}}{r}$$

and applying Poisson's formula, it is easily found that

$$\dot{\vec{p}} = 2(\vec{v} \cdot \vec{R}) \, \vec{r} - (\vec{r} \cdot \vec{R}) \, \vec{v} - (\vec{r} \cdot \vec{v}) \, \vec{R} \tag{12}$$

Now, in an equatorial system

$$\vec{i} = \begin{pmatrix} \cos\omega \cos\Omega + \sin\omega \sin\Omega \cos I \\ \cos\omega \sin\Omega + \sin\omega \cos\Omega \cos I \\ \sin\omega \sin I \end{pmatrix}$$

Therefore

$$\dot{\vec{p}} = \frac{d}{dt} (\mu \, e \, \vec{i}) = \mu \left[\dot{e} \, \vec{i} + e \, \frac{\partial \vec{i}}{\partial \Omega} \, \dot{\Omega} + e \, \frac{\partial \vec{i}}{\partial \omega} \, \dot{\omega} + e \, \frac{\partial \vec{i}}{\partial I} \, \dot{I} \right]$$

From $\dot{\vec{p}} \cdot \vec{i} = G \, \mu \, \dot{e}$, we obtain (10) in a new form

$$\dot{e} = \frac{\dot{\vec{p}} \cdot \vec{i}}{\mu} \tag{13}$$

and from (12)

$$\mu \dot{e} = 2(\vec{v} \cdot \vec{R}) \, (\vec{r} \cdot \vec{i}) - (\vec{r} \cdot \vec{R}) \, (\vec{r} \cdot \vec{i}) - (\vec{r} \cdot \vec{v}) \, (\vec{R} \cdot \vec{i}) \tag{14}$$

From

$$\dot{\vec{p}} \cdot \frac{\partial \vec{i}}{\partial \omega} = \mu e \, (\cos I \, \dot{\Omega} + \dot{\omega})$$

we deduce

$$\dot{\omega} = - \cos I \, \dot{\Omega} + \frac{\dot{\vec{p}} \cdot \frac{\partial \vec{i}}{\partial \omega}}{\mu e} \tag{15}$$

Therefore

$$\dot{\omega} = - \frac{\cos I}{2C \sin^2 I}\left[\vec{r}, \vec{R}, \frac{\partial \vec{k}}{\partial \Omega}\right] + \frac{1}{\mu e}\left[2(\vec{v} \cdot \vec{R})\left(\vec{r} \cdot \frac{\partial \vec{i}}{\partial \omega}\right) + \right.$$

$$\left. - (\vec{r} \cdot \vec{R})\left(\vec{v} \cdot \frac{\partial \vec{i}}{\partial \omega}\right) - (\vec{r} \cdot \vec{v})\left(\vec{R} \cdot \frac{\partial \vec{i}}{\partial \omega}\right)\right] \tag{16}$$

In resume, equations (6), (8), (9), (10) or (14), (16) give the variational equations for a, Ω, I, e, ω.

4. Time of Pericenter

$$\tau = t - \mu^{-1/2} a^{3/2} (E - e \sin E)$$

Applying Poisson's formula

$$\dot{\tau} = - \mu^{1/2} \operatorname{grad}_{\vec{v}} \left[a^{3/2} (E - e \sin E)\right] \cdot \vec{R} \tag{17}$$

Now

$$\operatorname{grad}_{\vec{v}} a = \frac{2a^2}{\mu} \vec{v}$$

$$\operatorname{grad}_{\vec{v}} E = \frac{1}{e \sin E}\left\{\frac{4 \cos E}{\mu^2 e}\left[(hr^2 + C^2)\vec{v} - h(\vec{r} \cdot \vec{v})\vec{r}\right] - \frac{2r}{\mu}\vec{v}\right\}$$

$$\operatorname{grad}_{\vec{v}} e = \frac{4}{\mu^2 e}\left[(hr^2 + C^2)\vec{v} - h(\vec{r} \cdot \vec{v})\vec{r}\right]$$

and these relations enable us to compute $\dot{\tau}$, from (17). Alternatively we may compute $\dot{\sigma}$ where $\sigma = \ell - nt$.

SUMMARY

The equations for the variation of constants have been deduced in a more direct way, without the necessity of computing Lagrange brackets. On the other hand it seems that the present method

is suitable for any integral and gives the direct variation of a desired element by the use of a single equation. Furtheremore, it gives the variations of the integrals of motion in an elegant way. It is hoped that the present method simplifies the deduction to a great extent.

The method is essentially equivalent to a method suggested to the author by Dr. C. J. Cohen, where Poisson's variational equation is applied directly to the elliptic elements, considered as functions of position and velocity.

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